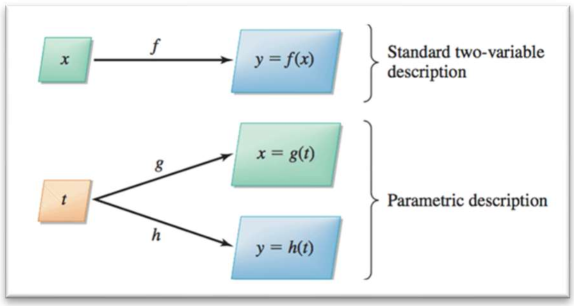
MSLC Workshop Series
Math 1172 - Workshop 4
Vector-Valued Functions

Parametric Curves in 2-D:



Parametric curves are curves given by $x = g(t), y = h(t)$ for some independent variable t, usually thought of as time. So all the points on the curve can be given by $(x, y) = (g(t), h(t))$.
Caution: These curves need not be graphs of functions.

<u>Here are some parametric curves you should be able to recognize:</u>

A line segment from $(\boldsymbol{x_1}, \boldsymbol{y_1})$ to $(\boldsymbol{x_2}, \boldsymbol{y_2})$

$$x = x_1 + (x_2 - x_1)t \text{ and } y = y_1 + (y_2 - y_1)t, \quad 0 \le t \le 1$$

A circle centered at $(\boldsymbol{x_0}, \boldsymbol{y_0})$ with a radius $\boldsymbol{a}$.

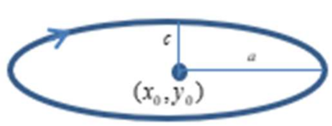
$$\boldsymbol{x = x_0 + a\cos(bt), \; y = y_0 + a\sin(bt)}$$

The circle is generated clockwise if $\mathrm{b} > 0$ *and counterclockwise if* $\mathrm{b} < 0$.

An ellipse centered at $(\boldsymbol{x_0}, \boldsymbol{y_0})$:

$$\boldsymbol{x = x_0 + a\cos(bt), \; y = y_0 + c\sin(bt)}$$

The circle is generated clockwise if $\mathrm{b} > 0$ *and counterclockwise if* $\mathrm{b} < 0$.



It is sometimes possible to **eliminate the parameter** by solving one equation for t and plugging it into the other equation. This will give the same curve, but you will lose the information about the direction and speed given by the parameter t .

You can turn functions $y = f(x)$ into parametric curves simply by letting $x = t, y = f(t)$.

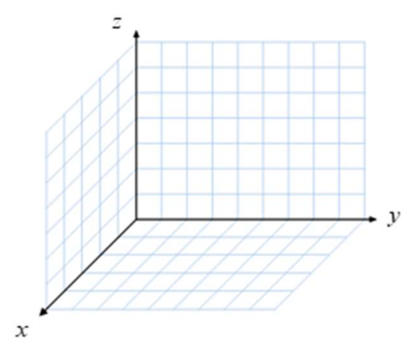
Parametric Curves in 3-D:

This is essentially exactly the same as 2-D but you get a third equation:

$$x = f(t), y = g(t), z = h(t)$$

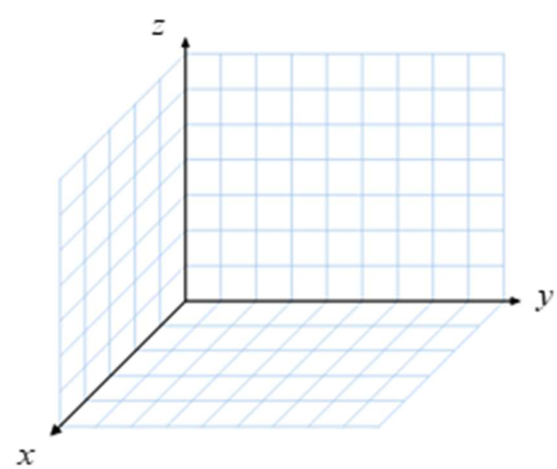
which gives you a point in 3-space $(x, y, z) = (f(t), g(t), h(t))$.

Eliminating the parameter of a 3-D curve will not give you a nice, single equation like in 2-D. For many purposes, parametric descriptions are the most natural way to describe higher dimensional curves.



<u>Example:</u> What does the following parametric equation look like? Describe its properties.

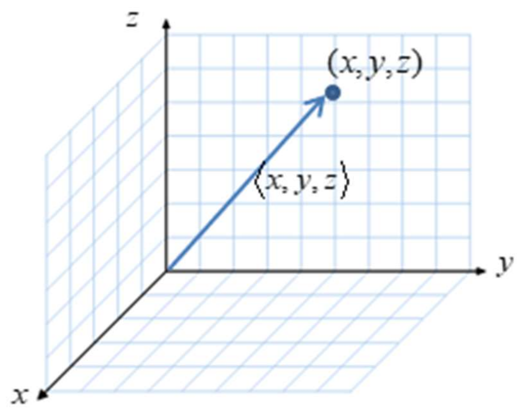
$$x = 3 + 2\cos t\,, y = 4 + 2\sin t\,, z = 2t, \quad 0 \le t \le 6\pi$$



Vectors:

In 3-D, it is often more helpful to talk about vectors instead of points. A **vector** is an object with a **magnitude** (length) and a **direction**. We draw it as an arrow.

It does not matter where a vector is sitting in space, but if a vector $\langle x, y, z\rangle$ has its tail at the origin, then its head will be at the point (x, y, z). (People often interchange these two related but distinct concepts.)



Vector Operations Examples:

(Table on last page of handout):

1. Find the Magnitude of $\mathbf{v} = \langle 3,7,2\rangle$

2. Simplify the following: $\mathbf{v} = \langle 3,7,2\rangle, 5\mathbf{v} = ?$

3. $\langle 3,7,2\rangle + 4\langle 1,2,3\rangle$

4. Write $\mathbf{v} = \langle 3,7,2\rangle$ in terms of $\mathbf{i,j,k}$

5. $\langle 3,7,2\rangle \cdot \langle 1,2,3\rangle$

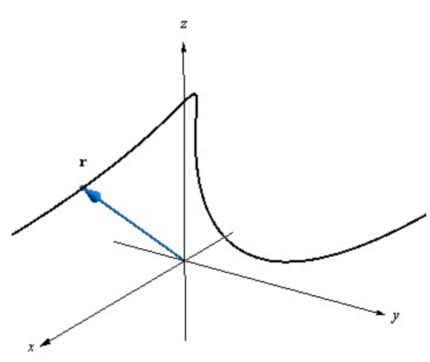
6. $\mathbf{v} = \langle 3,7,2 \rangle, \mathbf{u} = \langle 1,2,3 \rangle, \text{proj}_{\mathbf{v}}\mathbf{u} = ?$

7. $\mathbf{v} = \langle 3,7,2 \rangle, \mathbf{u} = \langle 1,2,3 \rangle, \mathbf{v} \times \mathbf{u} = ?$

Vector-Valued Functions:

A vector-valued function is essentially a 3-D parameterization where we think of the output as a vector instead of a point: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

As *t* varies, the tail of the vector stays at the origin and the head of the vector traces out the 3-D parametric curve.



Equation of a Line

An **equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$, or

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle, \quad \text{for } -\infty < t < \infty.$$

Equivalently, the parametric equations of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for } -\infty < t < \infty$$

Example 1: Find the vector-valued function for the line which passes through the point $(1,2,3)$ in the direction $\langle 4,5,6\rangle$.

Example 2: Find the vector-valued function for the line which passes through the points $(1,2,3)$ and $(4,5,6)$.

Calculus of Vector-Valued Functions:

In general, it is very difficult to say anything about vector-valued functions without calculus. Thankfully, calculus on vector-valued functions is computationally very straightforward.

Limits:

DEFINITION: Limit of a Vector-Valued Function

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t\to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

Computationally, this means you can just take the limit of each component of the vector:

$$\lim_{t\to a} \boldsymbol{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \right\rangle$$

Example: Find the limit of $\mathbf{r}(t) = \langle 5t, e^{3t}, t^2 + 11 \rangle$ as $t \to 0$

Continuity:

A vector-valued function $\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$ is continuous at *a* if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$. This just means that $\mathbf{r}(t)$ is continuous at *a* if and only if $x(t), y(t)$, and $z(t)$ are all continuous at *a.*

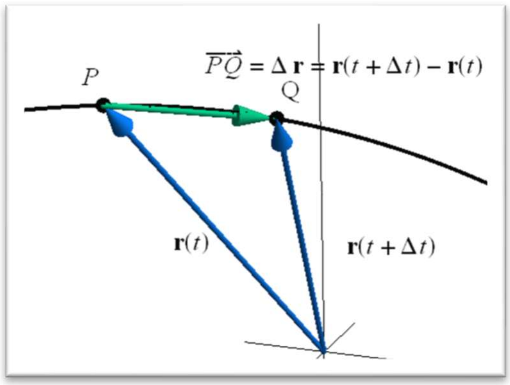
Example: Find the values of *t* where the following vector-valued function is not continuous.

$$\boldsymbol{r}(t) = \left\langle \frac{5}{t-3}, e^t, \tan t \right\rangle \qquad 0 \le t \le \pi$$

Derivatives:

We define the derivative of a vector-valued function to be:

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$



DEFINITION: Derivative and Tangent Vector

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions on (a, b). Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on (a, b) and

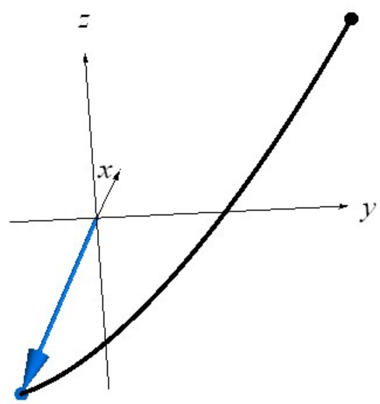
$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

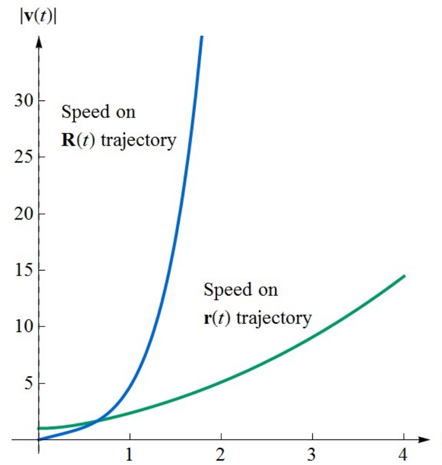
Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** (or velocity vector) at the point corresponding to $\mathbf{r}(t)$.

Example 1: Find the derivative of $\boldsymbol{r}(t) = \left\langle t, t^2 - 4, \frac{1}{4}t^3 - 8 \right\rangle$.

Example 2: Find the derivative of $\boldsymbol{R}(t) = \left\langle t^2, t^4 - 4, \frac{1}{4}t^6 - 8 \right\rangle$.

If $\boldsymbol{r}(t)$ gives the position of a particle in space at time t, then the derivative of $\boldsymbol{r}(t)$ gives the **velocity** of the particle and magnitude of the derivative gives the **speed** of the particle.





Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

1. $\frac{d}{dt}(\mathbf{c}) = 0$ — Constant Rule
2. $\frac{d}{dt}\big(\mathbf{u}(t) + \mathbf{v}(t)\big) = \mathbf{u}'(t) + \mathbf{v}'(t)$ — Sum Rule
3. $\frac{d}{dt}\big(f(t)\mathbf{u}(t)\big) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ — Product Rule
4. $\frac{d}{dt}(\mathbf{u}(f(t)) \;=\; \mathbf{u}'(f(t))f'(t)$ — Chain Rule
5. $\frac{d}{dt}\big(\mathbf{u}(t) \cdot \mathbf{v}(t)\big) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ — Dot Product Rule
6. $\frac{d}{dt}\big(\mathbf{u}(t) \times \mathbf{v}(t)\big) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ — Cross Product Rule

Find the derivatives of the following vector-valued functions.

1. $(5t^3 + \ln t)\boldsymbol{r}(3t + 11)$. Give the answer in terms of the vector $\mathbf{r}$ and t.

2. $\boldsymbol{u}(t) \cdot (\boldsymbol{v}(t) \times \boldsymbol{w}(t))$

3. $t\langle 4t, \ln t, 3\rangle + 7\langle 5,6,1\rangle$

Integrals:

An indefinite integral is just an anti-derivative. Since the derivative for vector-valued functions is just the same as taking the derivative of each component, the indefinite integral of a vector-valued function is just taking the indefinite integral of each component.

$$\int \boldsymbol{r}(t)dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle$$

DEFINITION: Indefinite Integral of a Vector-Valued Function

Let $\mathbf{r} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ be a vector function and let $\mathbf{R} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinite integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(\boldsymbol{t})\boldsymbol{dt} = \mathbf{R}(\boldsymbol{t}) + \mathbf{C}$$

where $\mathbf{C}$ is an arbitrary constant vector.

Example: Find the indefinite integral of the following vector-valued function.

$$\int (e^t\boldsymbol{i} + 12\boldsymbol{j} + cos(3t)\boldsymbol{k})\, dt$$

We can define a definite integral of a vector-valued function similarly.

DEFINITION: Definite Integral of a Vector-Valued Function

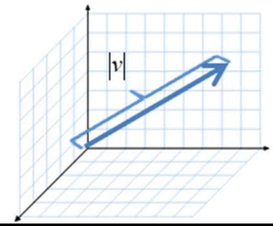
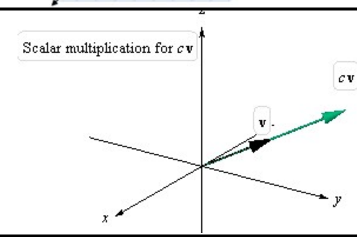
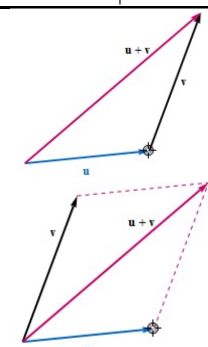
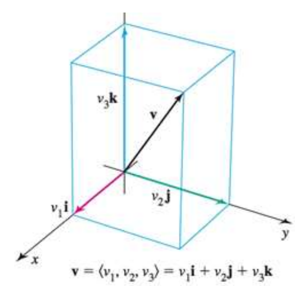
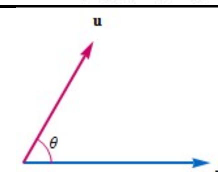
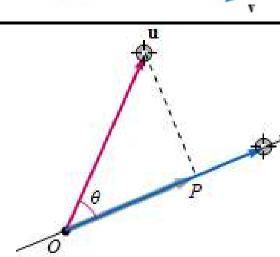
Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are integrable on the interval $[a, b]$.

$$\int_a^b \mathbf{r}(t)dt = \left[\int_a^b f(t)dt\right]\mathbf{i} + \left[\int_a^b g(t)dt\right]\mathbf{j} + \left[\int_a^b h(t)dt\right]\mathbf{k}$$

Example: Find the indefinite integral of the following vector-valued function.

$$\int_3^5 \left((4 + 7t)\boldsymbol{i} + \boldsymbol{j} - \sqrt{t}\boldsymbol{k}\right) dt$$

Important Vector Operations: $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle, \boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$

Term	Description	Formula	Graph
Magnitude	length	$\lvert\mathbf{v}\rvert = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}$	
Multiplying by a scalar	stretching	$c\mathbf{v} = \langle cv_1, cv_2, cv_3 \rangle$	
Adding	Putting end to tail -or- Diagonal of the parallelogram formed by the two vectors	$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$	
Basis Vectors	you can write all the others in terms of 3 main vectors	$\mathbf{i} = \langle 1,0,0 \rangle$ $\mathbf{j} = \langle 0,1,0 \rangle$ $\mathbf{k} = \langle 0,0,1 \rangle$ $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$	
Dot Product	Multiplying two vectors to get a scalar. Allows you to find angles between vectors.	$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$ $= \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert \cos\theta$	
Projection	Allows you to project **u** onto **v**	$\text{proj}_{\mathbf{v}}\mathbf{u} = \lvert\mathbf{u}\rvert \cos\theta \left(\frac{\mathbf{v}}{\lvert\mathbf{v}\rvert}\right)$ $= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}$	
Cross Product	Multiplying two vectors to get a vector. The magnitude of the cross product is the area of the parallelogram.	$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ $\lvert\mathbf{u} \times \mathbf{v}\rvert = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert \sin\theta$