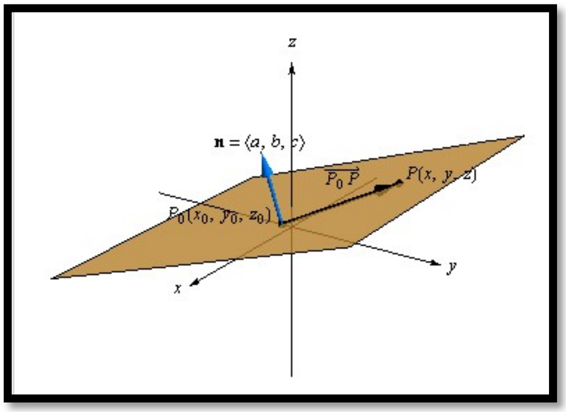
MSLC Workshop Series
Math 1172
Planes and Surfaces

I. Planes

A plane is like an infinite piece of paper in 3-space.

A plane can be determined if you know a **point** on the plane and a **normal vector** to the plane.

Compare this to finding the equation of a line in 2-space. You need a point to tell you the "height" and a slope or normal vector to tell you the "slant".



General Equation of a Plane in R^3

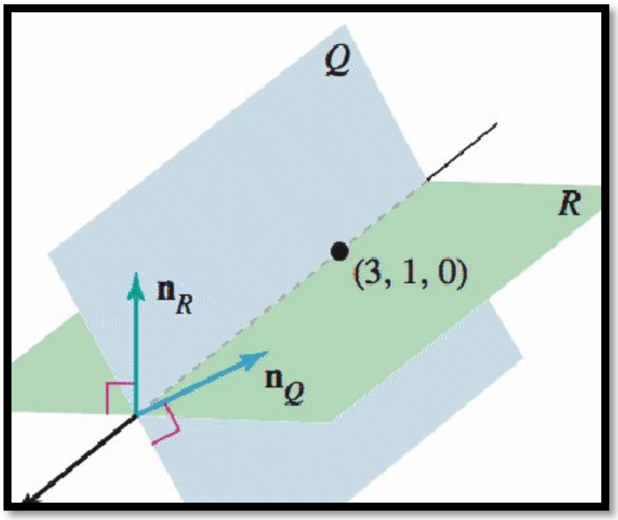
The plane passing through the point $P_0 = (x_0, y_0, z_0)$ with a normal vector $\mathbf{n} = \langle a, b, c\rangle$ is described by the equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$.

Example 1: Find the equation of the plane that contains the point (7,8,9) and is perpendicular to the vector <1,2,3>.

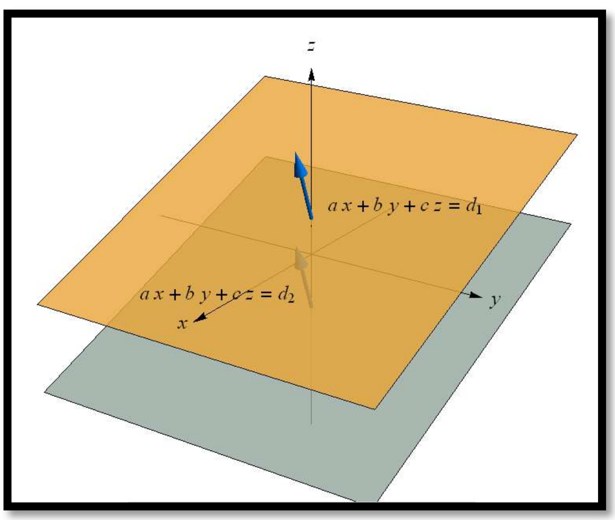
Example 2: Find the plane that contains the points (3,2,1), (4,5,6), and (7,8,9).

II. Intersection of Planes:

When two planes distinct intersect, they intersect in a line.



Sometimes two planes are parallel and they do not intersect.



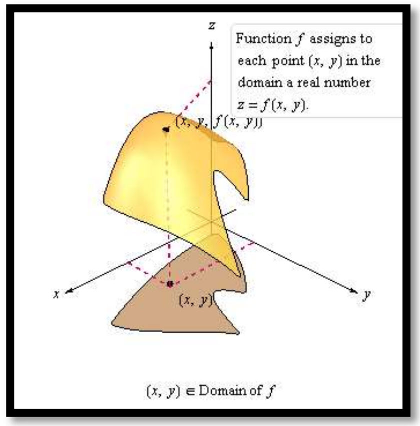
Example 1: Determine if the following planes intersect, and if they intersect, determine the line of intersection.

$$3x + 2y - 5z = 9,\ \ 2x - 3y + z = 2$$

Example 2: Determine if the following planes intersect, and if they intersect, determine the line of intersection.

$$3x + 2y - 5z = 9, \qquad 6x + 4y - 10z = 2$$

III. Surfaces

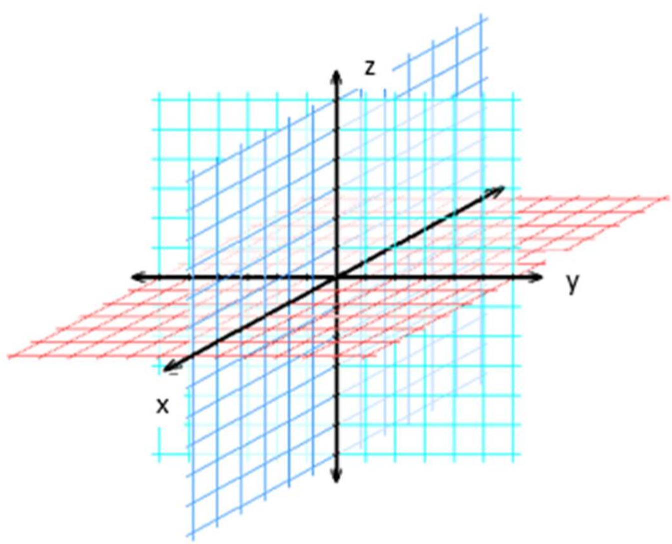


Surfaces are 2-dimensional objects. Anything that looks "locally" like a plane is called a surface. This means, if you are a tiny bug living on this surface, you think it's a plane. For example, the surface of the earth is a surface. We think it looks flat, but we know it's really the surface of a sphere.

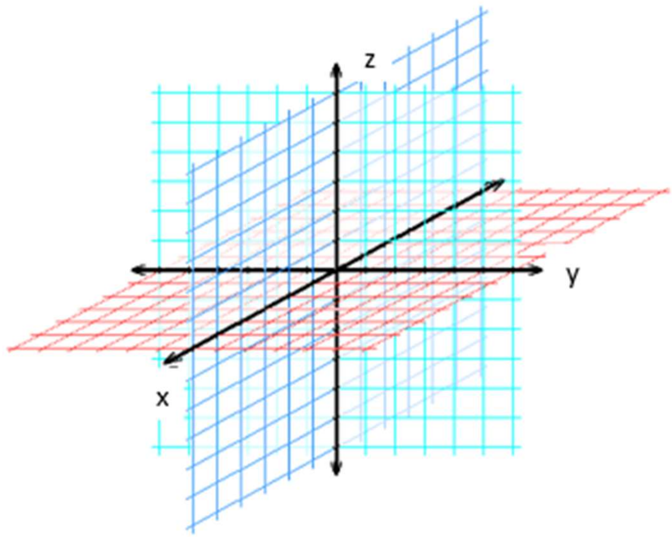
Some surfaces are graphs of equations in two variables which look like $z = f(x, y)$. We call these **functions of two variables** if they pass the vertical (i.e. parallel to z) line test.

Surfaces can be very difficult to draw or even visualize. Some are very easy, though.

Example 1: Sketch $z = 6$. This is a plane.



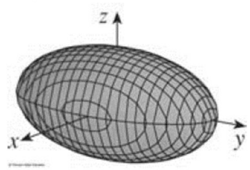
Example 2: Sketch $z = y^2$. This is a cylinder.



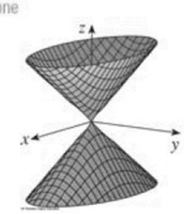
IV: Quadric Surfaces

There are some surfaces that are used so often that you really just need to memorize them.

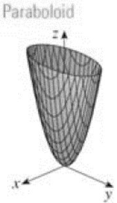
ELLIPSOID

Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.

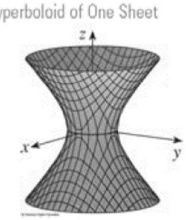
CONE

Surface	Equation
Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

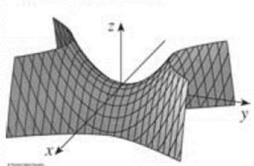
ELLIPTIC PARABOLOID

Surface	Equation
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.

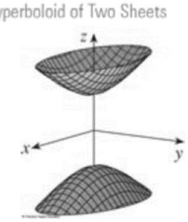
HYPERBOLOID OF ONE SHEET

Surface	Equation
Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.

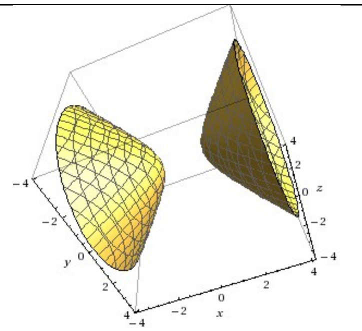
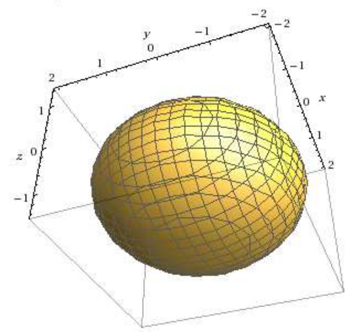
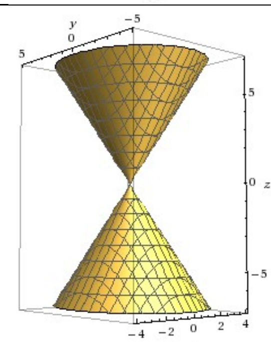
HYPERBOLIC PARABOLOID

Surface	Equation
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.

HYPERBOLOID OF TWO SHEETS

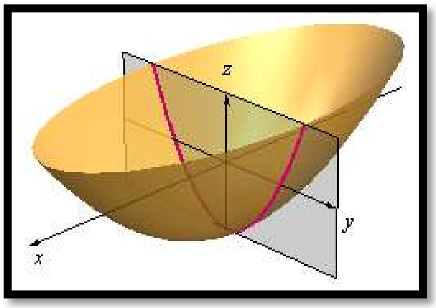
Surface	Equation
Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Example: Match the functions with their graphs and names. Choose one item per column.

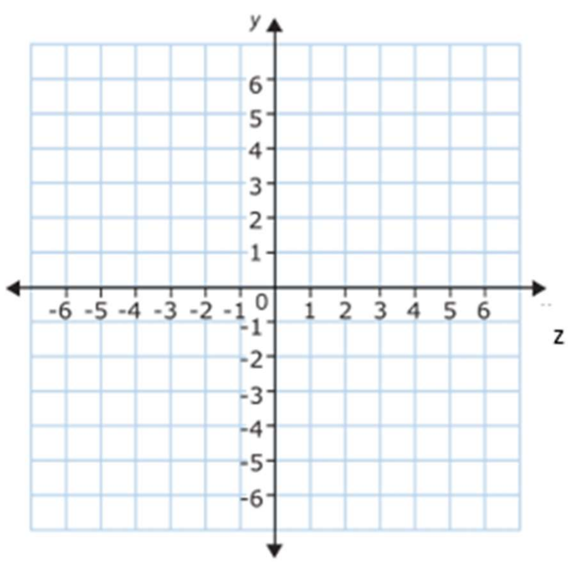
Function	**Picture**	**Name**
$z^2 = 3x^2 + 2y^2$		Hyperboloid of two sheets
$x^2 - y^2 - z^2 = 1$		Cone
$x^2 + y^2 + 2z^2 = 4$		Ellipsoid

V: Traces

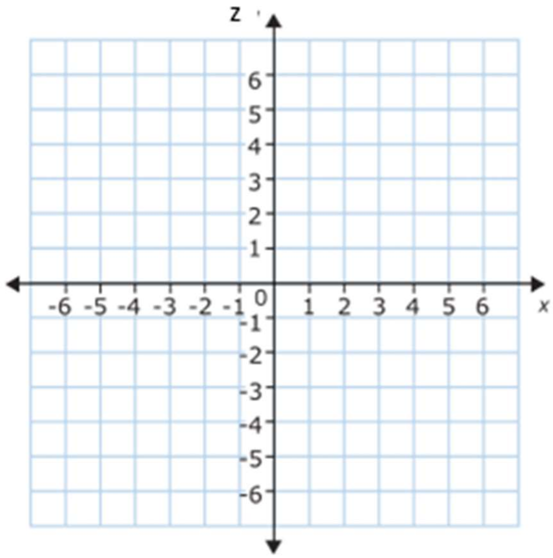
To find out more about quadric surfaces or other unknown surfaces, we use various techniques to try to look at portions of these surfaces in 2-D. One such technique is called **traces**. For this technique, you intersect the surface with a plane, usually a $x = a$, $y = b$, or $z = c$, and look at the resulting 2-D curve.



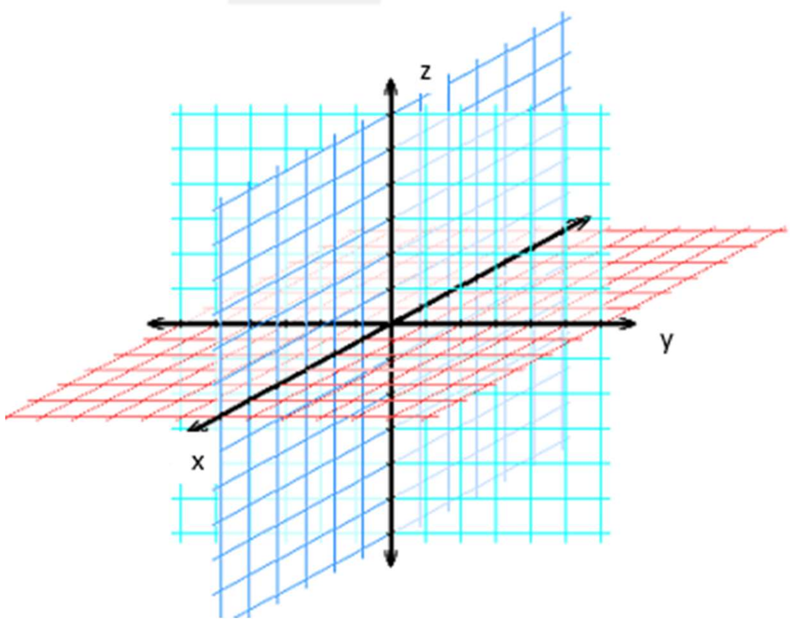
Example 1a: Find the $x = 0$ trace for $y = \frac{z^2}{4} + \frac{x^2}{4}$. Sketch this curve, and compare this to your sketch on the previous page.



Example 1b: Find the trace when y = 4 for $y = \frac{z^2}{4} + \frac{x^2}{4}$. Sketch this curve, and compare this to your sketch on the previous page.



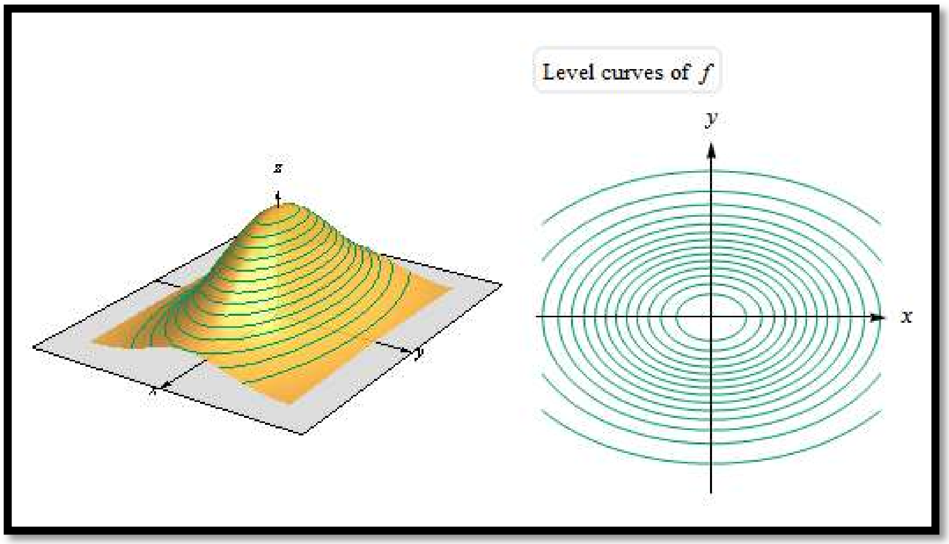
Example 1c: Sketch $y = \frac{z^2}{4} + \frac{x^2}{4}$



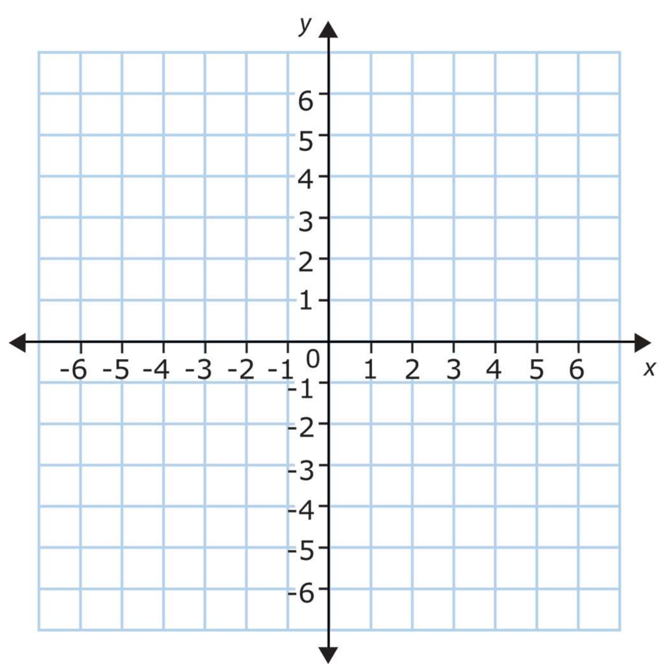
VI: Level Curves

Another technique for determining how a surface looks or behaves is by using level curves. This is used is used when the surface is a function of two variables, $z = f(x, y)$.

Level curves are essentially a series of traces made with planes parallel to the xy-plane ($z = c$). These traces are all graphed on the same set of xy-axis and are used to visualize how the function changes as we change the height, z. A common example of level curves in use are elevation maps.



Example 1: Draw the level curves for $f(x, y) = y - x^2 - 1$ for $f(x, y) = -2, 0,$ and 2.



Example 2: Draw the level curves for $f(x, y) = e^{-x^2-y^2}$ for $f(x, y) = 0.1, 0.3,$ and 0.7.