MSLC Workshop Series
Calculus 2
Taylor Series

I. Concept:

RECALL: A polynomial is a function that looks like $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

Wouldn't it be nice if all functions could be represented as polynomials?

Unfortunately, that is impossible. What we can do is:

1. Approximate all differentiable functions with tangent lines
2. Approximate a function with n derivatives at a point with a degree n polynomial.
3. If a function is infinitely differentiable at a point, we take the limit of the sequence of approximating polynomials to get an "infinite degree" polynomial.

Definition: We call this limit of a sequence of polynomials a **Power Series**. That is, a power series is an infinity series of the form $\sum_{k=0}^{\infty} c_k(x-a)^k$. Here, the point x=a is where we are choosing to base our approximation. It is called the center of the power series. Given a particular function, if it is power series centered at x=a that is equal to the given function on an interval at x=a, we call that power a **Taylor Series**.

II. Approximating a Function with a Polynomial:

Example:

1. Find a line that approximates $f(x) = e^x$ near the point $x = 0$.

2. Find a parabola approximates $f(x) = e^x$ near the point $x = 0$.

In general, we can say:

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

3. Find an nth degree polynomial that approximates $f(x) = e^x$near the point $x = 0$.

III. Taylor Series: The "infinitely long" polynomial *(it's really a limit of polynomials as n→∞)*

Say we wanted to approximate the value of $\int_0^1 e^{x^2}\,dx$ with as much accuracy as we would like. Then it wouldn't be good enough to approximate with any finite degree polynomial because that would not allow us to control the error. We need something that is actually equal to e^{x^2} on the interval from [0,1].

> **<u>Definition:</u>**
> The **Taylor Series** of a function, $f(x)$, is the *only possible* power series representation for the function $f(x)$, and the Taylor series **centered at *a*** has the form:
> $$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{2\cdot 3}(x-a)^3 + \ldots.$$
> A Taylor Series centered at 0 is called a **Maclaurin Series**.

Example: Find the Taylor Series of e^x centered at $x = 0$.

Since a Taylor Series is a Power Series, it has an interval of convergence. To find the radius of convergence, use the Ratio Test.[1]

> **<u>Ratio Test:</u>**
> 1. If $0 \leq \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
> 2. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
> 3. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive.

Example: Find the interval of convergence for the Power Series of e^x centered at $x = 0$

[1] *Note: This will not tell you what happens on the endpoints of the interval of convergence. In Math 1172, you do not need to worry about the endpoints. In Math 1152, you should use other convergence tests to determine if the endpoints converge or diverge.*

IV. Maclaurin Series: Taylor Series centered at x=0

You are allowed and encouraged to memorize and use the following Maclaurin Series

Table of Maclaurin Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

You can make new Maclaurin Series from these Maclaurin Series by using the rules for combining power series:

Combining Power Series:

Suppose the power series $\sum_{k=0}^{\infty} c_k x^k \to f(x)$ on the interval I and $\sum_{k=0}^{\infty} d_k x^k \to g(x)$ on the interval I

1. **Sum and Difference:** $\sum_{k=0}^{\infty} (c_k \pm d_k) x^k \to f(x) \pm g(x)$ on the interval I
2. **Multiplication by a Power:** $x^m \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^{k+m} \to x^m f(x)$ on the interval I, assuming $k+m \ge 0$ for all k
3. **Composition:** $\sum_{k=0}^{\infty} c_k (bx^m)^k = \sum_{k=0}^{\infty} c_k b^k x^{km} \to f(bx^m)$ where m is a positive integer, b Is a real number, bx^m is in the interval I

Example: Find the Maclaurin Series of e^{x^2}. Find the interval of convergence for this series.

Extra Practice: Find the Maclaurin Series of the following functions and their radii of convergence.

1. $f(x) = \tan^{-1}(x^3)$

2. $f(x) = \frac{7x^6}{6+18x}$

VI. Differentiating and Integrating Power Series

One of the big reasons engineers and scientists like Taylor Series is because they are so easy to integrate and differentiate. You just integrate or differentiate **term-by-term**.

Differentiating Power Series: $\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right]' = \sum_{n=0}^{\infty} c_n \left[(x-a)^n\right]' = \sum_{n=1}^{\infty} c_n \cdot n(x-a)^{n-1}$

Integrating Power Series: $\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx = \sum_{n=0}^{\infty} c_n \int \left[(x-a)^n\right] dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$

Example: $\int \tan^{-1}(x^3)\, dx$

You Try: $\frac{d}{dx}\frac{7x^6}{6+18x}$

VII. Finding Numerical Values:

You can also use Taylor Series to find numerical values to any desired level of accuracy. We've already done this when we found a decimal approximate for e, but now we can do it for more complicated "numbers".

1. Find $\int_0^1 e^{x^2}\,dx$

2. Find an infinite series that converges to $\tan^{-1}\left(\frac{1}{2}\right)$.