MSLC Workshop Series

Calculus II

Integration Techniques

You have seen several basic integration formulas. You will need to have all of these forms *memorized*, since they will form a basis for working with all the other integration techniques.

Properties of Integrals:

$\int kf(u)du = k\int f(u)du$	$\int [f(u) \pm g(u)]du = \int f(u)du \pm \int g(u)du$
$\int_a^a f(x)dx = 0$	$\int_a^b f(x)dx = -\int_b^a f(x)dx$
$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$	$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$
$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if f(x) is even	$\int_{-a}^a f(x)dx = 0$ if f(x) is odd
$\int_a^b g(f(x))f'(x)dx = \int_{f(a)}^{f(b)} g(u)du$	$\int udv = uv - \int vdu$

Integration Rules:

$\int du = u + C$ $\int u^n du = \frac{u^{n+1}}{n+1} + C$ $\int \frac{du}{u} = ln\|u\| + C$ $\int e^u du = e^u + C$ $\int a^u du = \frac{1}{ln\, a}a^u + C$	$\int sin\, u\, du = -cos\, u + C$ $\int cos\, u\, du = sin\, u + C$ $\int sec^2 u\, du = tan\, u + C$ $\int csc^2 u\, du = -cot\, u + C$ $\int csc\, u\, cot\, u\, du = -csc\, u + C$ $\int sec\, u\, tan\, u\, du = sec\, u + C$	$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$ $\int \frac{du}{\sqrt{a^2+u^2}} = arcsin\left(\frac{u}{a}\right) + C$ $\int \frac{du}{u\sqrt{u^2+a^2}} = \frac{1}{a}\backslash arcsec\left(\frac{\|u\|}{a}\right) + C$

Trig Formulas:

$sin^2(x) = \frac{1}{2}(1 - cos(2x))$	$tan\, x = \frac{sin\, x}{cos\, x}$	$sec\, x = \frac{1}{cos\, x}$	$cos(-x) = cos(x)$	$sin^2(x) + cos^2(x) = 1$
$cos^2(x) = \frac{1}{2}(1 + cos(2x))$	$cot\, x = \frac{cos\, x}{sin\, x}$	$csc\, x = \frac{1}{sin\, x}$	$sin(-x) = -sin(x)$	$tan^2(x) + 1 = sec^2(x)$

For more help, visit http://mslc.osu.edu

First technique: Algebraic Manipulation

- Multiply by the Conjugate
- Factor
- Rationalize
- Other

Second technique: U-Substitution

- Let u=f(x)
- du=f'(x)dx
- Change everything in the integral from x's to u's, even the dx

Algebra and U-Sub: So how do you recognize when to do each of these techniques? Practice, Practice, Practice!! Evaluate the following integrals. All of them can be done by using algebraic manipulations and u-substitutions.

1. $\int \frac{4}{x^2+9} dx$

2. $\int \frac{4x}{x^2+9} dx$

3. $\int \frac{4x^2}{x^2+9} dx$

4. $\int \frac{x^3}{x^2+3} dx$

5. $\int \frac{1}{e^x+e^{-x}} dx$

6. $\int \frac{x}{\sqrt{x+3}} dx$

Integration by Parts: This is the analog of the product rule for derivatives. Integration by parts is a very powerful method, and is used extensively in many applications.

Formula for IBP: $\int u dv = uv - \int v du$

Knowing which function to call *u* and which to call *dv* takes some practice. Here is a general guide:

u ↑	**I**nverse Trig Function	$(sin^{-1}x, arccos\, x$,etc)
	Logarithmic Functions	$(log\, 3\, \mathrm{x}, ln(\mathrm{x}+1)$,etc)
	Algebraic Functions	$(x^3, x+5, 1/x$,etc)
	Trig Functions	$(sin(5\mathrm{x}), tan(\mathrm{x})$,etc)
↓ dv	**E**xponential Functions	$(e^{3x}, 5^{3x}$,etc)

Functions that appear at the top of the list are more like to be *u,* functions at the bottom of the list are more like to be *dv.*

1. $\int ln\, x\, dx$

2. $\int 3xe^{2x} dx$

3. $\int tan^{-1}(x) dx$

4. $\int x^3\, ln(x^4)\, dx$

5. $\int x^2 \cos x\, dx$

Trig Integrals:

Integrals involving sin(x) and cos(x):

1. If the power of the sine is odd and positive:
   Goal: $u = cos\, x$
   i. Save a $du = sin(x)\, dx$
   ii. Convert the remaining factors to $\cos(x)$ (using $sin^2 x = 1 - cos^2 x$.)

2. If the power of the cosine is odd and positive:
   Goal: $u = sin\, x$
   i. Save a $du = cos(x)\, dx$
   ii. Convert the remaining factors to $sin(x)$ (using $cos^2 x = 1 - sin^2 x$.)

3. If both $sin(x)$ and $cos(x)$ have even powers:
   Use the half angle identities:
   i. $sin^2(x) = \frac{1}{2}(1 - cos(2x))$
   ii. $cos^2(x) = \frac{1}{2}(1 + cos(2x))$

Integrals involving sec(x) and tan(x):

1. If the power of $sec(x)$ is even and positive:
   Goal: $u = tan\, x$
   i. Save a $du = sec^2(x)\, dx$
   ii. Convert the remaining factors to $tan(x)$ (using $sec^2 x = 1 + tan^2 x$.)

2. If the power of $tan(x)$ is odd and positive:
   Goal: $u = sec(x)$
   i. Save a $du = sec(x)\, tan(x)\, dx$
   ii. Convert the remaining factors to $sec(x)$ (using $sec^2(x) - 1 = tan^2(x)$.)

3. If there are no $sec(x)$ factors and the power on $tan(x)$ is even and positive:
   Use $\sec^2 x - 1 = \tan^2 x$ to convert one $\tan^2(x)$ to a $sec^2(x)$. Repeat if necessary.

4. If nothing else works, try converting everything to $sin(x)$ and $cos(x)$

Rules for sec(x) and tan(x) also work for csc(x) and cot(x) with appropriate negative signs.

Integrate the following trig integrals:

1. $\int sin^6(\theta)\, cos^5(\theta)\, d\theta$

2. $\int tan^2(x)\, sec^2(x)\, dx$

3. $\int sec^3(5x)\, tan(5x)\, dx$

4. $\int \frac{sin^3 x}{\sqrt{cos\, x}} dx$

5. $\int \frac{cot(x)}{csc(x)} dx$

Trigonometric Substitution – We love trig so much that we put it into integrals that didn't even have trig to begin with!!

We often use this when we have square roots that we can't work with using any other method. What we substitute depends on the form of the integrand:

Expression	Substitution	Simplification
$\sqrt{a^2 - u^2}$	$u = a\,sin\,\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\sqrt{a^2 - u^2} = a\,cos\,\theta$
$\sqrt{a^2 + u^2}$	$u = a\,tan\,\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\sqrt{a^2 + u^2} = a\,sec\,\theta$
$\sqrt{u^2 - a^2}$	$u = a\,sec\,\theta,\ 0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$	$\sqrt{u^2 - a^2} = a\,tan\,\theta$

After you make the substitution, simplify and integrate using the rules for trig integrals. Be sure to substitute back when you are finished (drawing a triangle may help.)

1. $\int \sqrt{16 - 9x^2}dx$

2. $\int \frac{1}{(9+x^2)^2}dx$

3. $\int_3^4 \frac{\sqrt{w^2-3}}{w}dw$

4. $\int \frac{x^3}{(4-x^2)^2}dx$

Partial Fractions – We are all familiar with the process of adding fractions. We must find a common denominator and then we just add the numerators. Sometimes, when we are integrating, it would be helpful to be able to "un-add" fractions. This is where partial fractions comes in. We want to determine which (simpler) fractions could be added together to get our original fraction.

For a proper rational function in reduced form, suppose the repeated linear factor $(x-r)^n$ appears in the denominator. Then, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)}+\frac{A_2}{(x-r)^2}+\frac{A_3}{(x-r)^3}+\ldots+\frac{A_n}{(x-r)^n}$$

where $A_1, A_2, A_3, \ldots, A_n$ are constants to be determined. (These constants are often labeled $A, B, C, D, etc.$)

For a proper rational function in reduced form, suppose the repeated irreducible quadratic factor $(ax^2+bx+c)^n$ appears in the denominator. Then, the partial fraction decomposition contains the sum

$$\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where $A_1, A_2, A_3, \ldots, A_n$ and $B_1, B_2, B_3, \ldots, B_n$ are constants to be determined. (These constants are often labeled $A, B, C, D, etc.$)

Examples:

1. Find the general form of the partial fraction decomposition. (Do not solve for any coefficients).

$$\frac{x^{18}-3x^2+1}{(x^2-3x)^2(x^2-x-6)^3(x^2+x+1)^2}$$

2. Evaluate the integral $\int \frac{x^3-5x^2-x-3}{x^4-1}\,dx$