MSLC Workshop Series
Calculus I
Sigma Notation and Riemann Sums

Sigma Notation:

Notation and Interpretation of $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \cdots + a_{n-1} + a_n$

- $\sum$ (capital Greek sigma, corresponds to the letter S) indicates that we are to **sum** numbers of the form indicated by the general term
- a_k is the **general term**, which determines what is being summed, and can be defined however we want but is usually a formula containing the index: $a_k = f(k)$
- *k* is called the **index**; we may use any letter for the index, typically we use i, j, k, l, m, and n as indices
- The index runs through the positive integers, starting with the number below the $\sum$ (in this case 1) and ending with the integer above the $\sum$ (in this case *n*)
- The sum on the right hand side is the **expanded form**. (The $\cdots$ contains all the terms I was too lazy to write.)
- The letter below the sigma is the variable with respect to the sum. All other letters are **constants** with respect to the sum.

Example: $\sum_{i=4}^{9} i^2 =$

Special Sum Formulas

$\sum_{i=1}^{n} 1 = n$ $\quad\quad$ $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\quad\quad$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\quad\quad$ $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example: $\sum_{i=1}^{1000} i^2 =$

Properties of Sigma Algebra $\sum$ is an **operator** that represents summation, and its properties are similar to the properties of addition (note what properties are **not** mentioned here)

- Multiplication by a common constant (also called a ***scalar*** multiple) $\sum ca_k = c\sum a_k$
- Addition or Subtraction (this is also called the ***linearity*** property) $\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k$

Example: $\sum_{k=1}^{20} (3k^3 + 6k) =$

Riemann Sums: $R = \sum_k (\text{height of } k\text{th rectangle}) \cdot (\text{width of } k\text{th rectangle})$

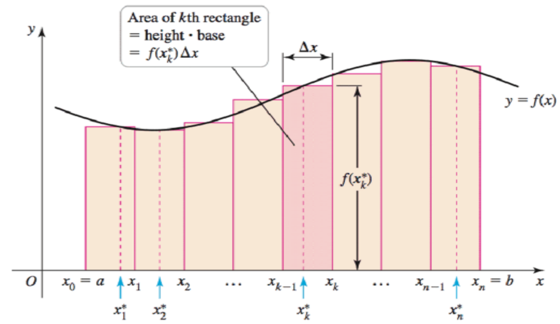


FIGURE 5.8

Definition of a Riemann Sum:
Consider a function $f(x)$ defined on a closed interval $[a,b]$, partitioned into n subintervals of equal width by means of points $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. On each subinterval $[x_{k-1}, x_k]$, pick an arbitrary point x_k^*. Then the Riemann sum for f corresponding to this partition is given by:

$$R = \sum_{k=1}^{n} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x$$

- **WIDTH:** Δx
  Since we partition the interval into evenly spaced partitions, we can calculate the width:

  $\Delta x = \dfrac{b-a}{n}$, where *n* is the number of partitions.

- **HEIGTH:** $f(x_k^*)$
  Also, we usually don't pick x_k^* arbitrarily. We use a rule to pick x_k^*. The most common rules to use are the Right Endpoint Rule, the Left Endpoint Rule, and the Midpoint Rule.

  The most common rules to use are:
  Right Endpoint Rule $x_k^* =$
  Left Endpoint Rule $x_k^* =$
  Midpoint Rule $x_k^* =$

- If we partition the interval into more and more rectangles with smaller and smaller widths, we get closer to the (signed) area trapped between the curve $y = f(x)$ and the x-axis.

  This is where **Sigma Notation** comes in because it becomes time consuming to add up all the terms when there are many, many rectangles.

Calculating A Riemann Sum

Using the **Right Endpoint Rule**, the Riemann sum becomes:

$$\sum_{k=1}^{n} f(a+k\Delta x)(\Delta x) = \sum_{k=1}^{n} \left(\tfrac{(b-a)}{n}\right) f\left(a+k\tfrac{(b-a)}{n}\right)$$

Using the **Left Endpoint Rule**, the Riemann sum becomes:

$$\sum_{k=1}^{n} f(a+(k-1)\Delta x)(\Delta x) = \sum_{k=1}^{n} \left(\tfrac{(b-a)}{n}\right) f\left(a+(k-1)\tfrac{(b-a)}{n}\right)$$

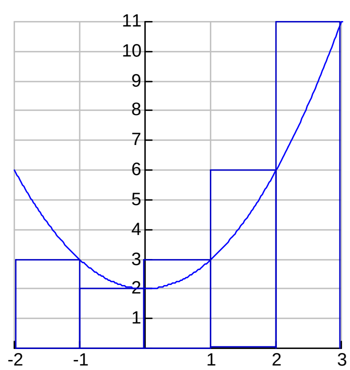
Using the **Midpoint Rule**, the Riemann sum becomes:

$$\sum_{k=1}^{n} f\left(a+\left(\tfrac{(k-1)+k}{2}\right)\Delta x\right)(\Delta x) = \sum_{k=1}^{n} \left(\tfrac{(b-a)}{n}\right) f\left(a+\left(\tfrac{(k-1)+k}{2}\right)\tfrac{(b-a)}{n}\right)$$

Example:

Estimate the area under $f(x) = x^2 + 2$ on the interval [-2, 3] using right Riemann Sums and 5 rectangles.

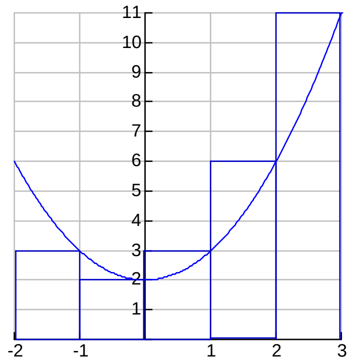
NOVICE (before Calculus):



Example:

Estimate the area under $f(x) = x^2 + 2$ on the interval [-2, 3] using right Riemann Sums and 5 rectangles.

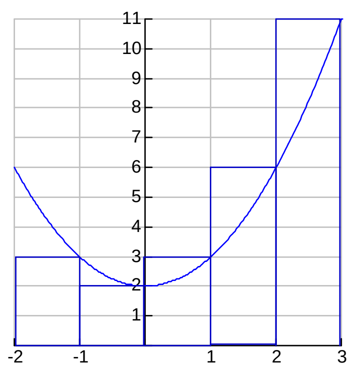
SEMI-PRO (beginning Calculus student):



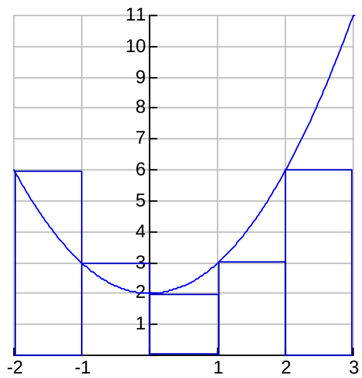
Example:

Estimate the area under $f(x) = x^2 + 2$ on the interval [-2, 3] using right Riemann Sums and 5 rectangles.

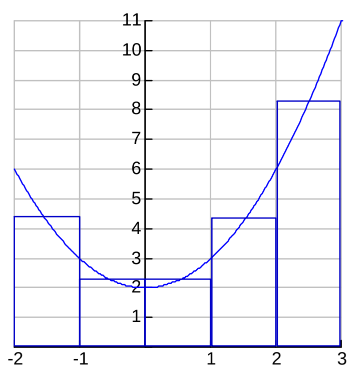
PRO (by test time):



Left Sum: Estimate the area under $f(x) = x^2 + 2$ on the interval [-2, 3] using left Riemann Sums and 5 rectangles.



Midpoint Sum: Estimate the area under $f(x) = x^2 + 2$ on the interval [-2, 3] using midpoint Riemann Sums and 5 rectangles.



Being More Accurate:

What if we want to get a better approximation than any of the above give us? More rectangles will give us less extra or unused area between the curve and the x-axis. Let's do a **right sum** for $\int_{-2}^{3}(x^2+2)dx$ using 5000 equal subintervals.

First calculate the width: $\Delta x =$

Then the x-value for the right endpoint of the *k*th rectangle is:

Thus the height of the *k*th rectangle is:

So the Riemann sum is

Now evaluate this sum using your knowledge of sigma algebra!