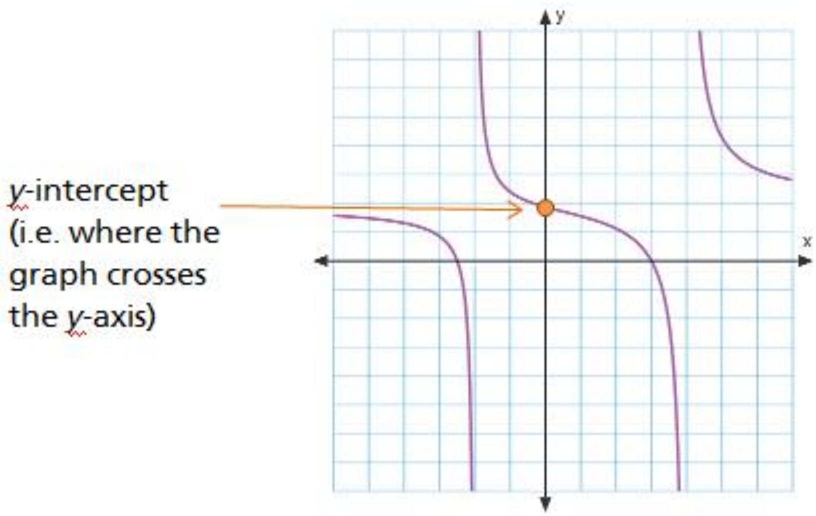
MSLC Workshop Series
Math 1148 – 1150 Workshop: Polynomial & Rational Functions

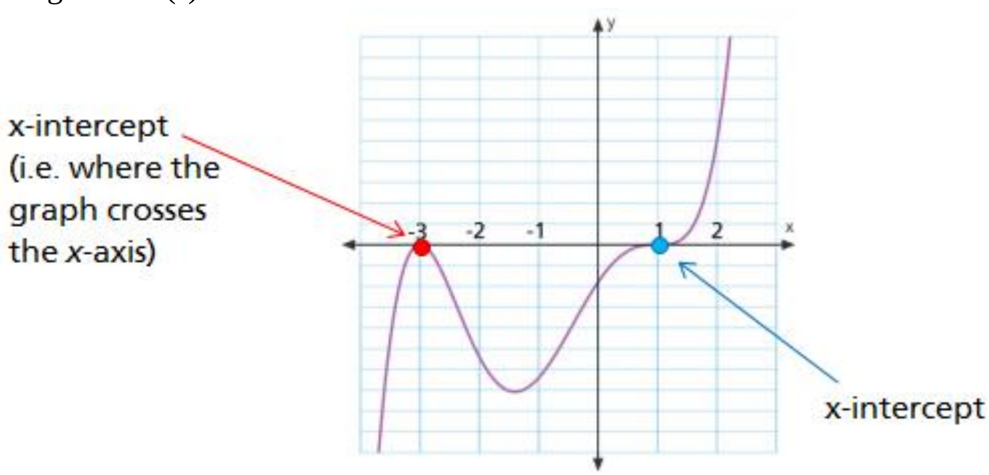
The goal of this workshop is to familiarize you with similarities and differences in both the graphing and expression of polynomial & rational functions.

We will start by looking at some of the similarities between these two types of functions.

1. **The *y*-intercept:** Both polynomial and rational functions can have a *y*-intercept. As the name implies this is where the graph of the function crosses the *y*-axis, and it is found by putting a zero in for *x* in the original function and solving for the corresponding *y*-value.



2. **The *x*-intercept(s):** Both polynomial and rational functions can have *x*-intercepts as well. This refers to points where the graph crosses the *x*-axis, and these are found by setting the function equal to zero and solving for the corresponding *x*-value(s).

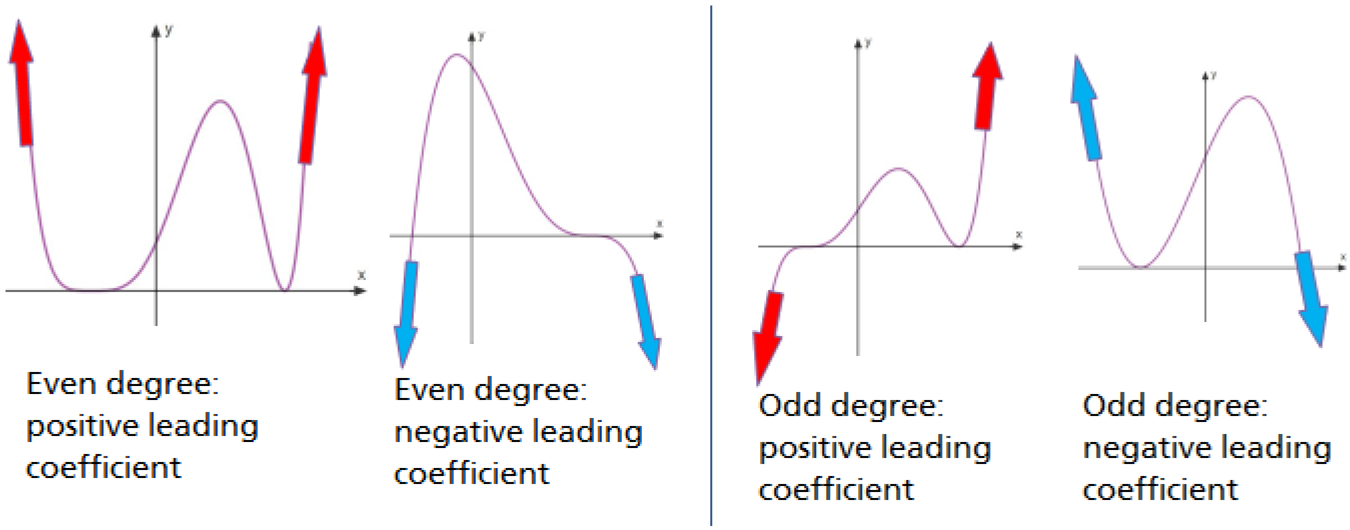


Features most relevant to polynomials.

3. **Degree:** This is an explicit feature of polynomials, but it is also used for analyzing rational functions.

The degree of a polynomial is the largest exponent present on variable in a single term. If multiple variables are present, the degree is determined by the largest sum of exponents on a single term.

- For polynomials the degree determines the end behavior of the graph. If the degree is an even number the graph will begin and end in the same location, either both up or both down (depending on the leading coefficient). If the degree is an odd number, the graph will begin and end in different locations, either starting down and ending up or starting up and ending down (depending on the leading coefficient).



For rational functions, we compare the degree of the polynomials in the numerator and denominator to determine if there will be a horizontal or slant asymptote, and this comparison also helps to determine where that asymptote is located.

4. **Factored Form:** Again, this typically is only specifically asked for when dealing with polynomials, but can also be used in analyzing rational functions.

a. The factored form of a polynomial has two possible variations.

i. The first uses real coefficients only and is found by breaking down the polynomial into a combination of linear or irreducible quadratic factors.

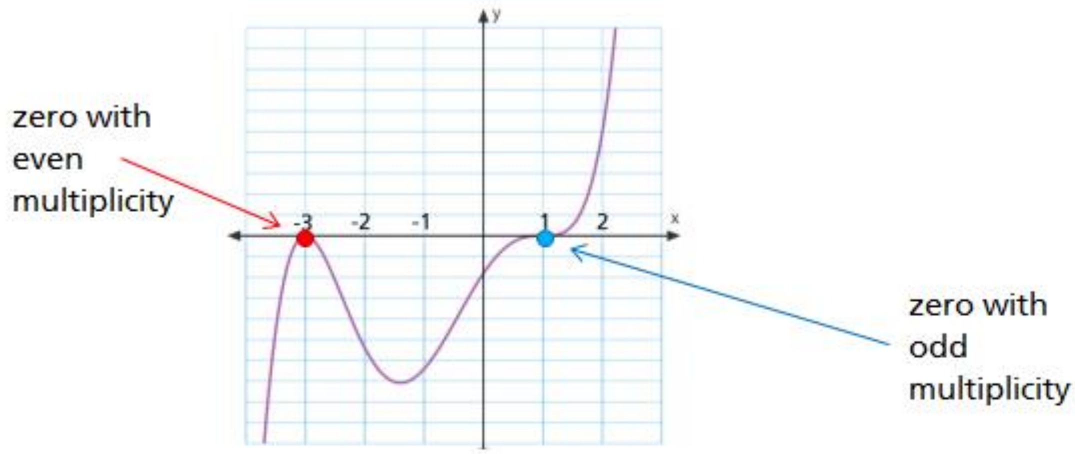
ii. The second uses both real and complex coefficients, which means that the polynomial can be broken down into linear factors only.

b. For rational functions, the numerator and denominator are typically only broken down into linear and irreducible quadratic factors (i.e. using real coefficients). This can allow us to more easily determine the zeros and vertical asymptotes of a rational function.

(*The factored form of a rational function also allows us to determine the completely simplified form, which is used for identifying removable discontinuities in the graph. But that is not something that is featured in this course.*)

5. **Multiplicity**: This is a property of the real zeros of the polynomial, and the polynomial must be in factored form to determine multiplicity. It refers to the exponent on an individual factor of a polynomial, and it is used to determine the behavior of the graph at each zero.

An odd multiplicity means the graph will cross the *x*-axis, and an even multiplicity will only touch the *x*-axis. *This feature isn't used for analyzing rational functions*.



<u>A few polynomial function problems</u>: Find the intercepts, degree, multiplicity of each zero, and leading coefficient of each of the following polynomials. Then use that information to sketch a graph of each polynomial.

$p(x)=x(x+7)(x-6)^2$

y-intercept of $p(x)$: zeros of $p(x)$ [and their multiplicities]:

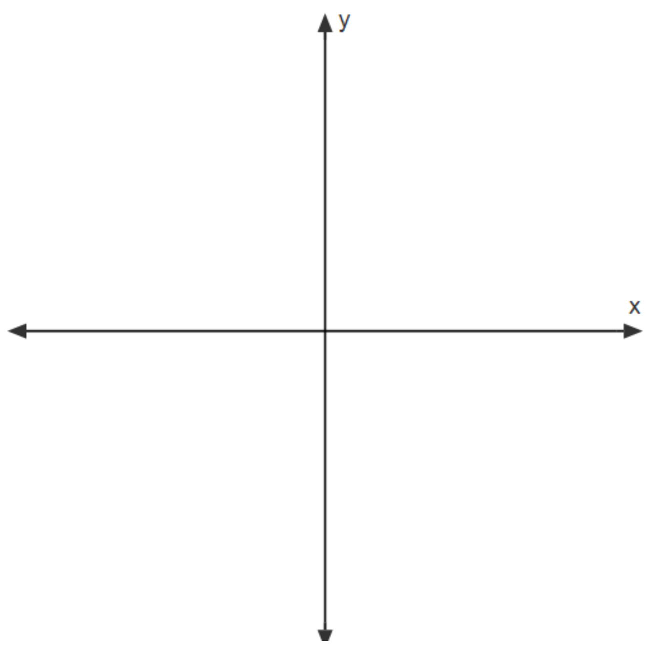
______________________ ____

degree of $p(x)$:

____

leading coefficient of $p(x)$:

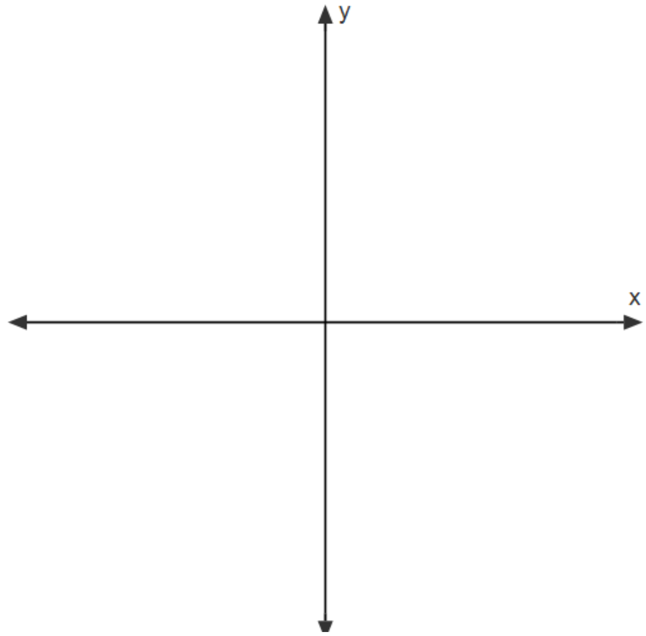
____



$P(x)=5x^5-50x^4-120x^3$

y-intercept of $P(x)$:

zeros of $P(x)$ [and their multiplicities]:



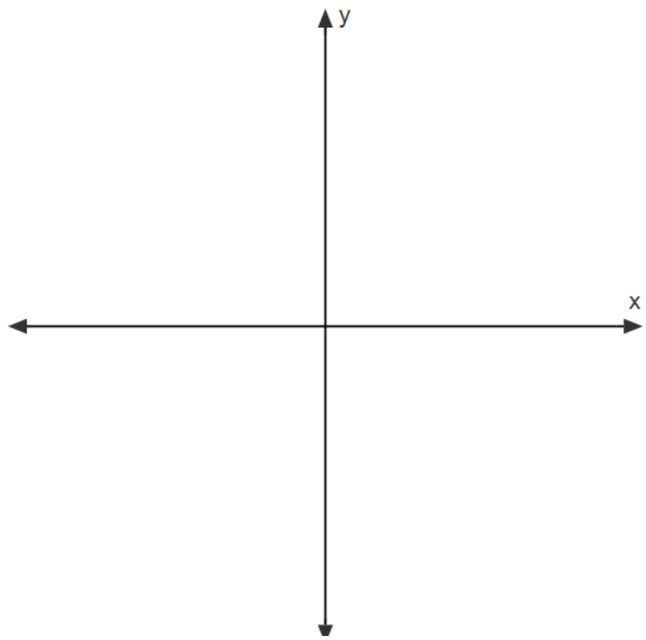
degree of $P(x)$:

leading coefficient of $P(x)$:

$Q(x)=(x-2)^2(x+1)^3(3x-8)$

y-intercept of $Q(x)$:

zeros of $Q(x)$ [and their multiplicities]:



degree of $Q(x)$:

leading coefficient of $Q(x)$:

Challenging Polynomial Problem: Factor the polynomial $h(x) = x^5 + 9x^3 - 8x^2 - 72$ into linear and irreducible quadratic factors with real coefficients. Then factor $h(x)$ into linear factors with complex coefficients.

Linear & Irreducible quadratic factors of $h(x)$ with real coefficients:

Linear factors of $h(x)$ with complex coefficients:

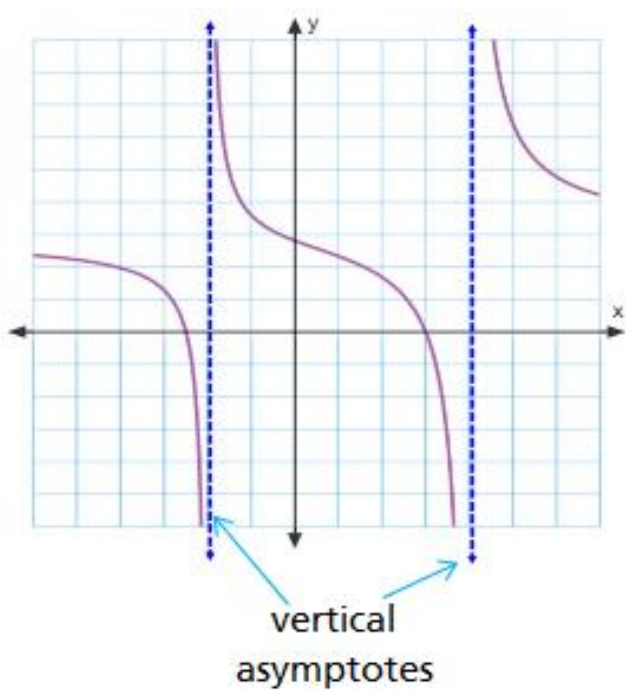
Features most relevant to rational functions.

6. **Vertical Asymptotes**: These occur in rational functions, but not in polynomials. They are found by setting the denominator of the rational function equal to zero and solving for *x*. These are lines and should be expressed as equations. **NOTE:** ***Rational functions WILL NEVER cross a vertical asymptote.***

Example:

The function $f(x) = \frac{6(x-6)(x+5)}{(x-8)(x+4)}$ has

vertical asymptotes at the lines $x = 8$ and $x = -4$.

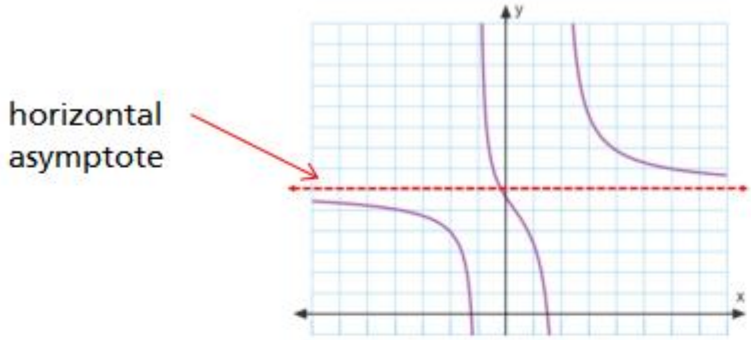


7. **Horizontal or Slant Asymptotes**: Horizontal and slant asymptotes can occur in rational functions, but not in polynomials. Determining if a rational functions has a horizontal or slant asymptotes is done by comparing the degree of the numerator and denominator.

<u>Horizontal Asymptote:</u>
If the degree of the numerator is less than or equal to the degree of the denominator, there will be a horizontal asymptote.

Example:

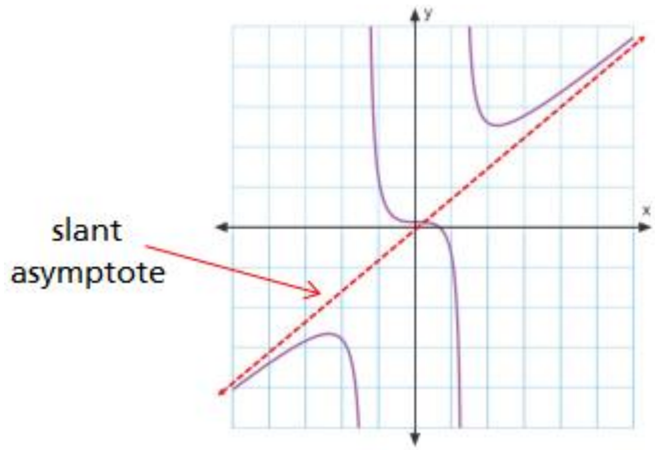
The function $f(x)=\frac{6(x-6)(x+5)}{(x-8)(x+4)}$ has a horizontal asymptote at the line $y=6$.



<u>Slant Asymptote:</u>
If the degree of the numerator is one degree larger than the degree of the denominator, there will be a slant asymptote.

Example:

The function $f(x)=\frac{x^3-64}{x^2-64}$ has a slant asymptote at the line $y=x$.

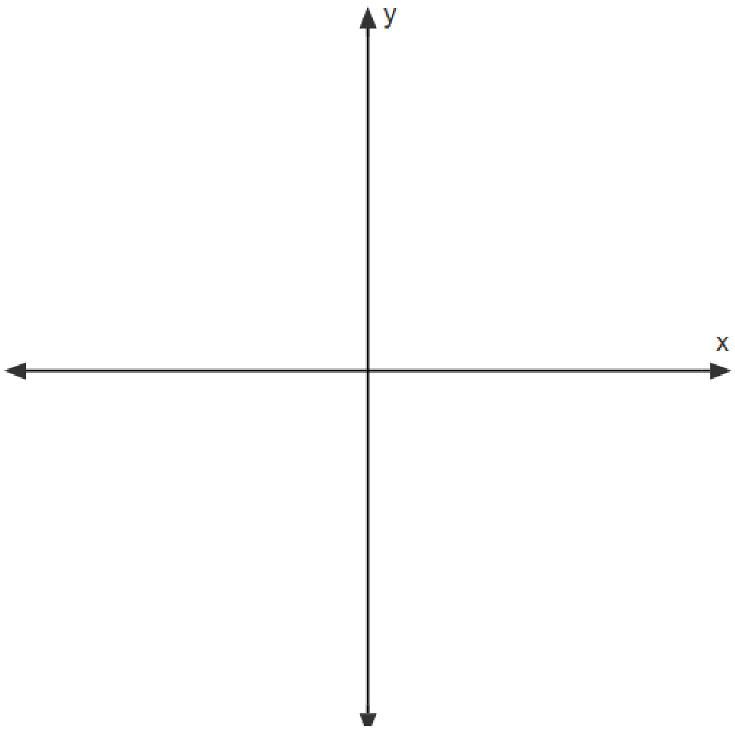


If the degree of the numerator is more than one degree larger than the denominator, the graph will follow another type of function (such as a parabola) asymptotically. *This situation is not discussed in this course*.

<u>NOTE:</u> ***It is possible for the graph of a rational function to cross horizontal and slant asymptotes. It is only the end behavior of the graph of a rational function that is determined by the horizontal or slant asymptote.***

A few rational function problems: Find the intercepts and asymptotes (vertical, horizontal, or slant) of each of the following rational functions. Then use that information to sketch a graph of each rational function.

$$r(x)=\frac{(x-3)(x+3)}{(x-4)(x+1)^2}$$



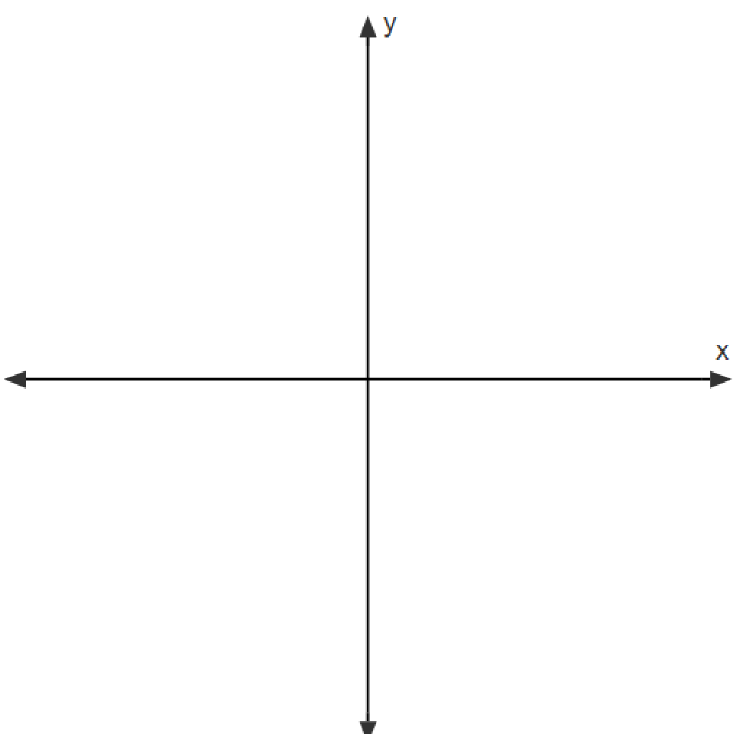
y-intercept of $r(x)$: ____________________

x-intercept(s) of $r(x)$: ____________________

vertical asymptote(s) of $r(x)$: ____________________

horizontal or slant asymptote of $r(x)$: ____________________

$$R(x)=\frac{(2x-1)(x+4)^2}{(x+2)(3x-5)^2}$$



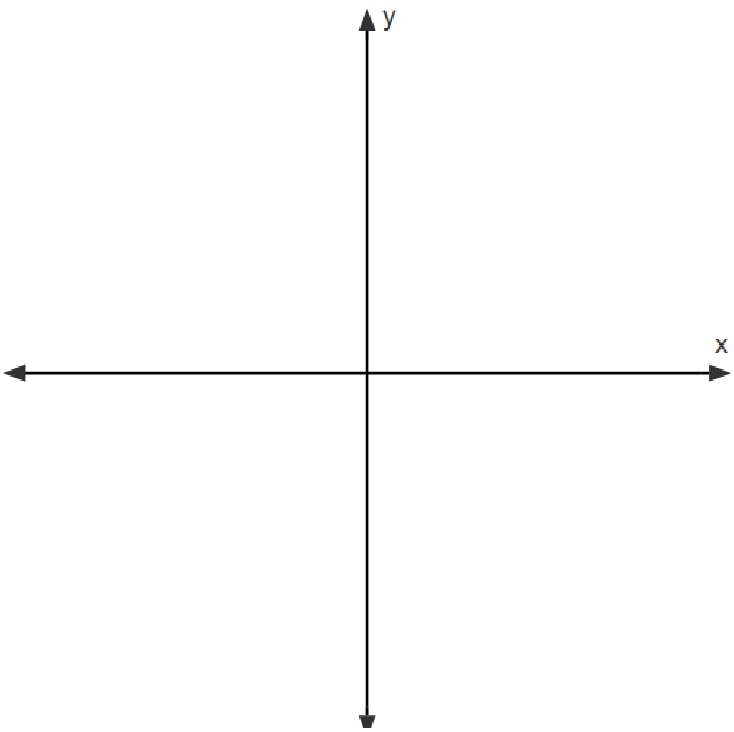
y-intercept of $R(x)$: ____________________

x-intercept(s) of $R(x)$: ____________________

vertical asymptote(s) of $R(x)$: ____________________

horizontal or slant asymptote of $R(x)$: ____________________

$q(x) = \dfrac{4x^2 - 16x}{2x + 8}$



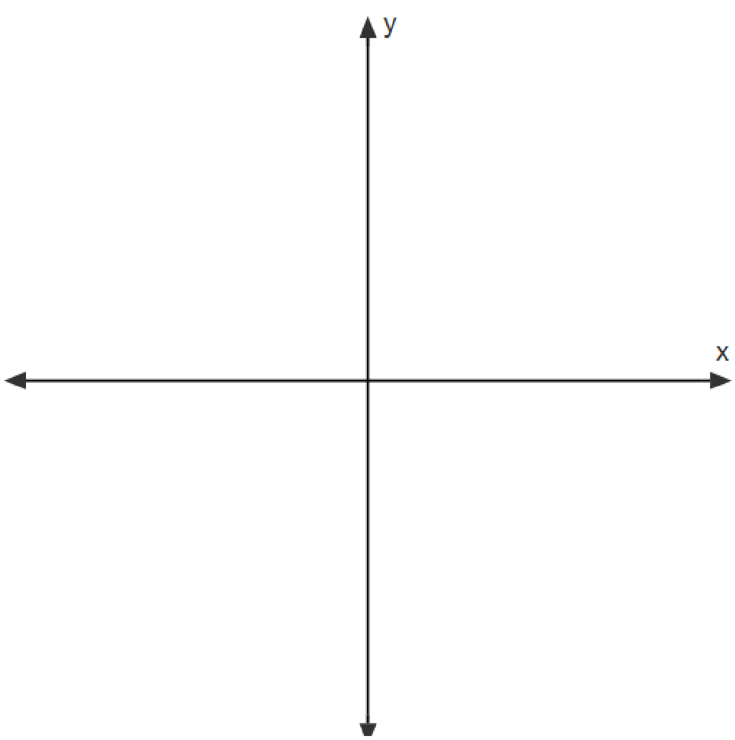
y-intercept of $q(x)$: ____________________

x-intercept(s) of $q(x)$: ____________________

vertical asymptote(s) of $q(x)$: ____________________

horizontal or slant asymptote of $q(x)$: ____________________

Challenging Case: Find the intercepts, asymptotes (vertical, horizontal, or slant), and location of any holes in the graph of $H(x) = \dfrac{5x^2 - 80}{4x^2 - 40x + 96}$. Then use that information to sketch a graph of $H(x)$.



y-intercept of $H(x)$: ____________________

x-intercept(s) of $H(x)$: ____________________

Holes in the graph of $H(x)$: ____________________

vertical asymptote(s) of $H(x)$: ____________________

horizontal or slant asymptote of $H(x)$: ____________________

Polynomial and Rational Functions: Important Feature Summary

Polynomial Functions	***Relevant Components***	***Rational Functions***
Found by setting $x = 0$ and solving for y	***y*-intercept**	Found by setting $x = 0$ and solving for y
Found by setting $y = 0$ and solving for x	***x*-intercept(s)**	Found by setting $y = 0$ and solving for x. Focus on the simplified form of the numerator, as it determines zeros.
The largest exponent present on a variable in a single term (or the largest sum of exponents on a single term if multiple variables are present).	**Degree**	Comparing the degree of the numerator to that of the denominator determines if there is a horizontal or slant asymptote, and the location of that asymptote.
Breaking polynomial down into component factors. Either linear and irreducible quadratic factors with real coefficients or linear factors with complex coefficients. Can be used to determine zeros and multiplicity of those zeros.	**Factored Form**	Breaking both numerator and denominator polynomials down into component factors typically linear and irreducible quadratic factors with real coefficients. This can be used to determine the zeros, the completely simplified form, and vertical asymptotes of the rational function.
Refers to the exponent on an individual factor. It's used to determine the graph's behavior at each zero. An odd multiplicity means that the graph will cross the *x*-axis at that zero, and an even multiplicity indicates the graph should touch the *x*-axis at that zero	**Multiplicity**	This isn't typically used for analyzing rational functions.
These don't occur in polynomial functions	**Vertical Asymptote(s)**	Found by setting the denominator of the completely simplified rational function equal to zero. Remember that asymptotes are lines and should be expressed as equations.
This is determined by the degree of the polynomial. If the degree is even, the graph will begin and end in the same direction (either both up or both down). If the degree is odd, the graph will begin in one direction and end in the other (i.e. start up and end down, or vice versa).	**End Behavior**	In rational functions this refers to what happens to the graph for very large (positive and negative) values of *x*. This refers to the effects of horizontal or slant asymptotes.
These don't occur in polynomial functions	**Horizontal/Slant Asymptotes**	Determined in part by comparing the degrees of the numerator and denominator. If the degree of the numerator is less than or equal to the degree of the denominator, there is a horizontal asymptote. If the degree of the numerator is one larger than the degree of the denominator, there will be a slant asymptote. Anything else will not result in either type of asymptote.