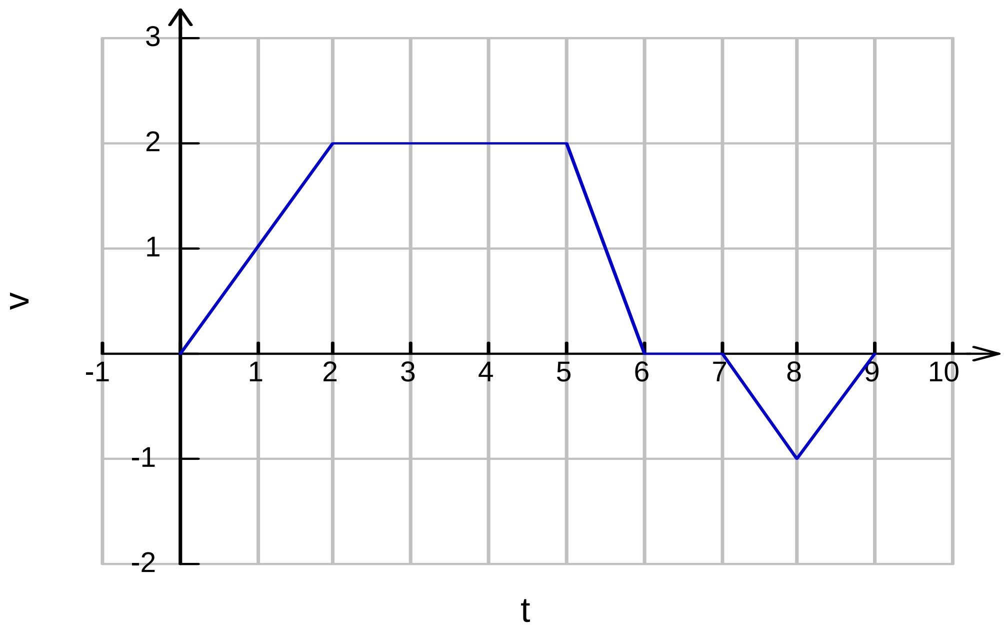
MSLC Workshop Series
Math 1151 – Workshop #7
The Fundamental Theorem of Calculus and U-Substitution

More on Area

Recall that the definite integral $\int_a^b f(x)\,dx$ gives us the ***signed*** area of the region trapped between the curve $y = f(x)$ and the *x*-axis on the interval [*a, b*]. This means that a positive sign is attached to areas above the *x*-axis, and a negative sign is attached to areas below the *x*-axis.

Let $f(x)$ be the function whose graph is shown to the right.

Using what you know from geometry, find $\int_0^9 f(x)\,dx$.



First Fundamental Theorem of Calculus

- $\int_a^b f(t)\,dt$ calculates the signed area under the curve $y = f(t)$ from a to b.

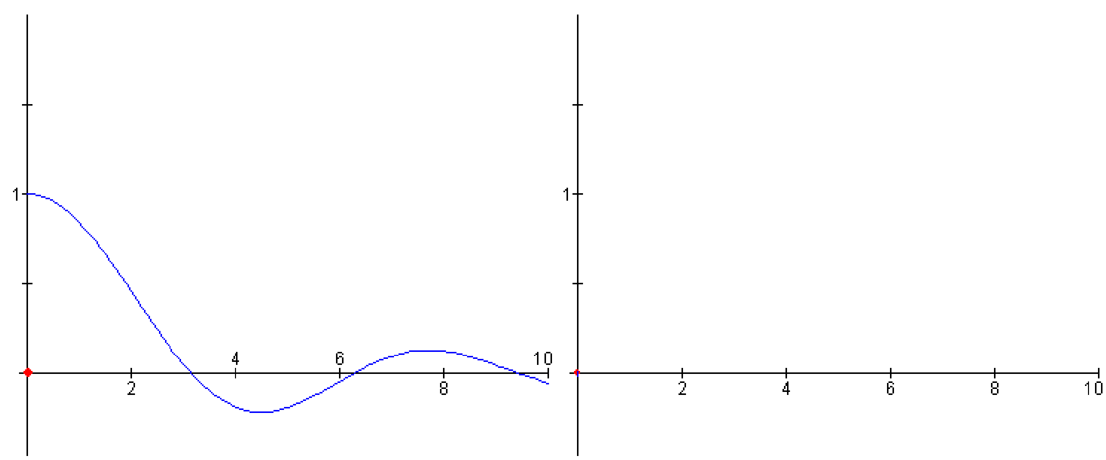
- Similarly, we can find the area under the curve between a and x (where x is a variable) with the integral $\int_a^x f(t)dt$.

- This integral can be thought of as a function of x! Let's call it:

$$F(x) = \int_a^x f(t)\,dt$$

(This function is called the accumulation function of the original function, f(x).)

Let's try to get an idea of what this function looks like in a specific case:
(See Demo: http://math.furman.edu/~dcs/java/ftc.html)



The rate of accumulation at *t* = *x* is equal to the value of the function being accumulated at *t* = *x*. This relationship is known as the **First Fundamental Theorem of Calculus**. That is:

$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, where $f(t)$ is a continuous function on [*a*, *x*].

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Examples: Find $F'(x)$.

1. $F(x) = \int_2^x \left(t^2 - \cos t + 3\right)dt$

2. $F(x) = \int_x^4 \frac{3}{t}\,dt$

3. $F(x) = \int_0^{x^2} t\cos(2t)\,dt$

4. $F(x) = \int_x^{x^3} \sin^2(t)\,dt$

Second Fundamental Theorem of Calculus

What if the function we are integrating doesn't lend itself to nice geometry calculation?

- We could use Riemann Sums

- OR -

- We could use the Second Fundamental Theorem, which states

$$\int_a^b f(x)\,dx = F(b) - F(a)$$, where F(x) is <u>any</u> antiderivative of f(x).

(This is much more efficient than using Riemann sums, assuming we can find an antiderivative of the function.)

Examples:

1. $\int_1^3 \left(2x^2 + x\right)dx$

2. $\int_0^{\pi/4} \sec^2(x)\,dx$

3. $\int_0^1 (x-2)(3-x)\,dx$

4. $\int_1^2 \frac{2x - \sqrt{x}}{x^2}\,dx$

U-Substitution – Undoing the Chain Rule

Suppose we want to find $\int \sin\left(x^4\right) 4x^3 dx$. We know how to find $\int \sin\left(\square\right) d\square$, so we are going to do a **CHANGE OF VARIABLE**.

$$\boxed{u = x^4}$$

But to do this, we also need to change the dx into a du.

$\frac{du}{dx} =$

Which gives us a new integral that we know how to solve:

CONVERTING ALL *x*'s to *u*'s

Most importantly, this change of variable will only work if we change EVERY x so that the ENTIRE integral is in terms of u.

When U-Substitution Fails:

$\int \sin\left(x^4\right) dx$

Making it match:

$\int x^3 \sin(x^4) dx$

Back Substitution:

$\int \frac{x}{x+1} dx$

Limits of Integration:

$\int_1^2 (x+3)^{60} dx$